Spring 2016 Topology Comprehensive Exam

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Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY which five you want us to grade; otherwise, if you do more than five problems, we will select five to grade, and they may not be the five that you want us to grade.

In the sequel, $\mathbb{R}$ denotes the real numbers and $\mathbb{N}$ denotes the positive integers. The usual topology on $\mathbb{R}$ will be denoted by τ_u. The closure of a subset A of a topological space (X, τ) will be denoted by $\text{cl}(A)$.

1. Prove that $(\mathbb{R}, \tau_u)$ is a connected topological space.

2. Let (X, τ) and (Y, σ) both be compact topological spaces. Prove that $X \times Y$ equipped with the product topology is compact (*hint*: it suffices to work with covers using basic open sets).

3. (a) What does it mean for a sequence $\langle x_n \rangle$ in a metric space (X, d) to be Cauchy?

   (b) Prove that a Cauchy sequence $\langle x_n \rangle$ with a convergent subsequence $\langle x_{n_k} \rangle$ must itself be convergent.

   (c) Show that if $\langle x_n \rangle$ is Cauchy, then $E := \{x_n : n \in \mathbb{N}\}$ is a bounded subset of X, that is, E is contained in some ball.

4. Let (X, τ) and (Y, σ) be topological spaces and let $f : X \to Y$ be continuous and onto.

   (a) Prove that if X is compact, then Y is compact.

   (b) Recall that $D \subseteq X$ is called *dense* if $\text{cl}(D) = X$. Show that if D is dense in X, then $f(D)$ is dense in Y.

5. (a) Prove that $\forall \alpha \in \mathbb{R}$, both $(-\infty, \alpha)$ and (α, ∞) belong the usual topology τ_u.

   (b) Suppose (X, τ) is a topological space and $f : X \to \mathbb{R}$. Show that f is continuous iff $\forall \alpha \in \mathbb{R}$, both $f^{-1}((-\infty, \alpha))$ and $f^{-1}((\alpha, \infty))$ are in τ. Here, $\mathbb{R}$ is equipped with τ_u.

6. Let C be a nonempty connected subspace of a topological space (X, τ). Show that if $C \subseteq D \subseteq \text{cl}(C)$, then D is connected as well.

7. Recall that (X, τ) is called *regular* if whenever $p \in V \in \tau$, there exists $W \in \tau$ with $p \in W \subseteq \text{cl}(W) \subseteq V$.

   (a) Prove that (X, τ) is regular iff whenever A is a nonempty closed subset and $p \notin A$, there exists disjoint V, W in τ with $p \in W$ and $A \subseteq V$.

   (b) Prove using part (a) that each compact Hausdorff space is regular.