Tiling With Trominoes

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Abstract

In this paper we consider tilings of $2\times n$ and $3\times n$ rectangles using trominoes of which there are two basic shapes, namely a 1×3 rectangle and an L-shaped figure. We will count how many ways the trominoes can be used to tile $2\times n$ and $3\times n$ rectangles and how many of each shape are used among all the tilings of a particular size rectangle.

1. Introduction

Solomon Golomb, in a 1953 talk at the Harvard Math Club, defined a class of geometric figures called *polyominoes,* namely, connected figures formed of congruent squares placed so each square shares one side with at least one other square. Dominoes, which use two squares, and tetrominoes (the *Tetris* pieces), which use four squares, are well known to game players. Golomb first published a paper about polyominoes in *The American Mathematical Monthly* [4]. Later, Martin Gardner popularized polyominoes in his *Scientific American* columns called "Mathematical Games" (see, for example, [2, 3]).

Many of the initial questions asked about polyominoes concern the number of *n-ominoes* (those formed from *n* squares), and what shapes can be tiled using just one of the polyominoes, possibly leaving one or two squares uncovered. In this paper we consider tilings using the 3-ominoes, or *trominoes*. We call the two types of trominoes *straights* and *Ls,* respectively, although others have named them as straight and right trominoes, respectively. Since there are only two trominoes, we count how many ways they can be used to tile $2\times n$ and $3\times n$ rectangles and how many of each shape are used among all the tilings of a particular size rectangle. Similar questions regarding Ls and squares are explored in [1]. Two books on polyominoes ([5, 6]) mention some results on tilings with trominoes. Both deal mainly with the questions of which figures can be covered with trominoes, or for which rectangles all but one or two squares can be so covered and where the missing squares can be located. Most recently, Jaime Rangel-Mondragón [8] used the computer algebra system Mathematica to create all polyominoes (as well as more general shapes) of a given size, and to create all tilings of a rectangular board with a given set of polyomino tiles. For example, he displays all the 41 tromino tilings of the 2×9 board. There is also a web site, *The Poly Pages* [7], which gives many examples of polyominoes and related shapes, as well as links to other pages studying these figures.

2. Notation and Basic Results

We will count the number of tilings, as well as the number of Ls and straights used in all the tilings of a given size. In order to do this, we will think of a tiling of size $m \times n$ as composed of a *basic block* (a tiling that cannot be split vertically into smaller rectangular tilings) of size $m \times k$, followed by a tiling of size $m \times (n-k)$. Note that *m* indicates the vertical size of the board, and that the second value (*n*, *k* or *n-k*) refers to the horizontal dimension. We will use the following notation:

$T(m,n)$ = number of tilings of size $m \times n$ with Ls and straights
$T_L(m,n)$ = number of Ls in all tilings of size $m \times n$
$T_S(m,n)$ = number of straights in all tilings of size $m \times n$
$B(m,n)$ = number of basic blocks of size $m \times n$
$B_L(m,n)$ = number of Ls in all basic blocks of size $m \times n$
$B_S(m,n)$ = number of straights in all basic blocks of size $m \times n$.

We also denote the generating function $\sum_{n=0}^{\infty} a(m,n)x^n$ for a sequence $\{a(m,n)\}_0^{\infty}$ by $G_{a(m)}(x)$. Since we can decompose any tiling into a basic block of some size on the left and a smaller tiling following it, we get this recursion:

$$T(m,n) = \sum_{k=1}^{n} B(m,n) \cdot T(m,n-k) \text{ for } n \geq 1, \tag{2.1}$$

where we define $T(m,0) = 1$ for any $m \geq 1$, in order to include the basic block of size $m \times n$ in the count.

Since the recursion for $T(m,n)$ is a convolution, the respective generating functions multiply (see, for example, [10], Section 2.2, Rule 3). Multiplying Equation (2.1) by x^n, summing over $n \geq 1$ and using the definition of the generating function, we obtain

$$G_{T(m)}(x) - 1 = G_{B(m)}(x) G_{T(m)}(x) \Rightarrow G_{T(m)}(x) = \frac{1}{G_{B(m)}(x)}. \tag{2.2}$$

We will also count the number of straights and Ls in all the tilings of an $m \times n$ board. Looking at the total area covered by all such tilings and splitting it up according to the areas covered by each type of tromino, we have the following equation:

$$m \cdot n \cdot T(m,n) = 3\,T_L(m,n) + 3\,T_S(m,n). \tag{2.3}$$

Therefore, we only have to count one of the two types of tiles. If we first look at counting straights, we get a recursion by creating the tilings from a basic block and a smaller tiling. For each such basic block, we get all the straights in the tilings of the smaller size, and then we get the number of straights in the basic block for each such smaller tiling. Thus, for $m \geq 2$ and $n \geq 1$,

$$T_S(m,n) = \sum_{k=1}^{n} B(m,k) \cdot T_S(m,n-k) + \sum_{k=1}^{n} B_S(m,k) \cdot T(m,n-k). \tag{2.4}$$

Again we encounter a convolution, and we obtain the results of Equation (2.5).

$$\begin{aligned} G_{T_S(m,n)}(x) &= G_{B(m)}(x) \cdot G_{T_S(m)}(x) + G_{B_S(m)}(x) \cdot G_{T(m)}(x) \\ \Rightarrow G_{T_S(m)}(x) &= \frac{G_{B_S(m)}(x) \cdot G_{T(m)}(x)}{1 - G_{B(m)}(x)}. \end{aligned} \tag{2.5}$$

The analogous formulas for the number of Ls are given by the following two equations.

$$T_L(m,n) = \sum_{k=1}^{n} B(m,k) \cdot T_L(m,n-k) + \sum_{k=1}^{n} B_L(m,k) \cdot T(m,n-k) \tag{2.6}$$

and

$$G_{T_L(m)}(x) = \frac{G_{B_L(m)}(x) \cdot G_{T(m)}(x)}{1 - G_{B(m)}(x)}. \tag{2.7}$$

These formulas allow us to reduce the counting of tilings to that of counting basic blocks.

3. Tiling $2 \times n$ Boards

Note that since both types of tiles cover an area of three units, it is only possible to tile $2 \times n$ boards when n is a multiple of 3, i.e., $T(2,n) = B(2,n) = 0$ unless $n \equiv 0 \pmod{3}$.

For $2 \times n$ basic blocks, the answer is rather simple. We start by looking at the number of basic blocks of size $2 \times n$. For $n = 3$, we obtain the basic block consisting of two straights and the two blocks that consist of two Ls each. For $n = 3k$, $k \geq 2$, we obtain two basic blocks, each with Ls at the end, and straights in between. The basic blocks for $n = 3$ and $n = 6$, as well as the general extension to sizes that are larger multiples of 3, are shown in Figure 1. To summarize, $B(2,3) = 3$, $B(2,3k) = 2$ for $k > 1$ and $B(2,n) = 0$ otherwise.

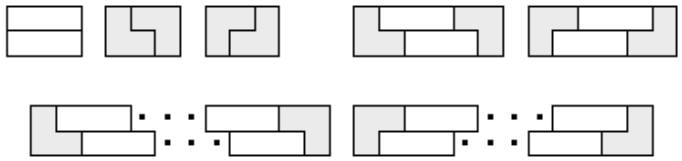


Figure 1. The basic blocks for tromino tilings of $2 \times n$ rectangles

The generating function for the number of basic blocks follows easily from the definition of the generating function and we obtain the following equation.

(3.1) $$G_{B(2)}(x) = 2\sum_{i=1}^{\infty} x^{3i} + x^3 = \frac{x^3(3-x^3)}{1-x^3}$$

Using the general form in Equation (2.2) and the generating function for the number of basic blocks, we get the generating function for the number of tilings:

(3.2) $$G_{T(2)}(x) = \frac{1}{1-G_{B(2)}(x)} = \frac{1-x^3}{1-4x^3+x^6}.$$

Table 1 below gives the number of tilings for $2 \times n$ rectangles.

n	0	3	6	9	12	15	18	21	24
$T(2,n)$	1	3	11	41	153	571	2131	7953	29681

Table 1: The number of tromino tilings of $2 \times n$ rectangles

This sequence appears in Sloane [9] as A001835, the number of ways of packing a $3 \times 2(n-1)$ rectangle with dominoes. This domino tiling is also discussed in [6, 8]. We can see the connection by noting that there are the same number of basic blocks for tiling a $3 \times 2n$ rectangle with dominoes as there are basic blocks for tiling a $2 \times 3n$ rectangle with trominoes, as shown in Figure 2.

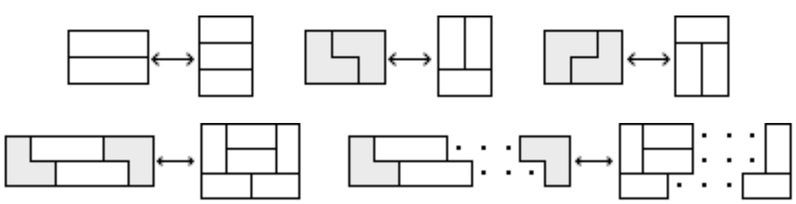


Figure 2. The correspondence between basic blocks for tromino and domino tilings

It is also possible to write recursive equations to generate the number of tilings of $2 \times 3k$ rectangles. In particular, using Equation (2.1), we find that

$$(3.3) \qquad \begin{aligned} T(2,3k) &= \sum_{i=1}^{n} B(2,3i) \cdot T(2,3(k-i)) \\ &= T(2,3(k-1)) + 2\sum_{i=1}^{n} T(2,3(k-i)). \end{aligned}$$

Substituting this equation into $T(2,3k) - T(2,3(k-1))$ and simplifying gives rise to the following recursive formula.

$$(3.4) \qquad T(2,3k) = 4T(2,3(k-1)) - T(2,3(k-2)),$$

with initial conditions $T(2,0) = 1$, $T(2,3) = 3$.

The corresponding characteristic equation is given by $x^2 - 4x + 1 = 0$, and the characteristic roots are $2+\sqrt{3}$ and $2-\sqrt{3}$. Consequently, we find that $T(2,3n) = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n$ and, using the initial conditions, we obtain $c_1 = \frac{1}{6}(3+\sqrt{3})$ and $c_2 = \frac{1}{6}(3-\sqrt{3})$. Since the resulting formula also gives the correct values for $n = 1$ and $n = 2$, we have proved the following theorem.

Theorem 3.1. *The number of tilings of* $2 \times 3k$ *rectangles with trominoes for* $k \geq 0$ *is given by*

$$(3.5) \qquad T(2,3k) = \frac{1}{6}(3+\sqrt{3}) \cdot (2+\sqrt{3})^k + \frac{1}{6}(3-\sqrt{3}) \cdot (2-\sqrt{3})^k,$$

with generating function

$$(3.6) \qquad G_{T(2)}(x) = g(x^3), \text{ where } g(x) = \frac{1-x}{1-4x+x^2}.$$

We next count the number of Ls and straights among all tromino tilings of $2 \times 3k$ rectangles, and derive explicit formulas as well as generating functions for both counts.

Theorem 3.2. *The number of Ls and straights in all tilings of* $2 \times 3k$ *rectangles with trominoes for* $k \geq 0$ *are given by*

$$T_L(2,3k) = \frac{1}{3\sqrt{3}}\left((2+\sqrt{3})^k - (2-\sqrt{3})^k\right) + \frac{k}{3}\left((1+\sqrt{3})\cdot(2+\sqrt{3})^k + (1-\sqrt{3})\cdot(2-\sqrt{3})^k\right) \tag{3.7}$$

and

$$T_S(2,3k) = \frac{1}{3\sqrt{3}}\left((2-\sqrt{3})^k - (2+\sqrt{3})^k\right) + \frac{2k}{3}\left((2+\sqrt{3})^k + (2-\sqrt{3})^k\right), \tag{3.8}$$

with generating functions

$$G_{T_L(2)}(x) = \frac{4x^3(1-x^3)}{(1-4x^3+x^6)^2} \quad \textit{and} \quad G_{T_S(2)}(x) = \frac{2x^3(1+x^6)}{(1-4x^3+x^6)^2}. \tag{3.9}$$

Proof: From Figure 1 we notice that each set of basic blocks of size $2\times 3k$ contains four Ls. Thus,

$$B_L(2,3k) = 4 \quad \text{for } k \ge 1 \;\Rightarrow\; G_{B_L(2)}(x) = \frac{4x^3}{1-x^3}, \tag{3.10}$$

and the generating function for the number of Ls in Equation (3.9) follows from Equation (2.7), together with Equations (3.1) and (3.2). To derive the explicit formula for the number of Ls, we use Equation (2.6) to obtain the following recurrence for the number of Ls, taking into account that the number of basic blocks and tilings is zero unless n is a multiple of 3:

$$\begin{aligned} T_L(2,3k) &= \sum_{i=1}^{k} B(2,3i)\cdot T_L(2,3(k-i)) + \sum_{i=1}^{k} B_L(2,3i)\cdot T(2,3(k-i)) \\ &= T_L(2,3(k-1)) + 2\sum_{i=1}^{k} T_L(2,3(k-i)) + 4\sum_{i=1}^{k} T(2,3(k-i)). \end{aligned} \tag{3.11}$$

Substituting Equation (3.11) into $T_L(2,3k) - T_L(2,3(k-1))$ and simplifying yields

$$T_L(2,3k) = 4\cdot T_L(2,3(k-1)) - T_L(2,3(k-2)) + 4\cdot T(2,3(k-i)), \tag{3.12}$$

a nonhomogeneous second order recurrence relation. The characteristic equation for the associated homogeneous relation is $x^2 - 4x + 1 = 0$, with roots $r_1 = 2+\sqrt{3}$ and $r_2 = 2-\sqrt{3}$. Thus, the homogeneous and particular solutions are of the form $a^{(h)} = c_1\cdot r_1^k + c_2\cdot r_2^k$ and $a^{(p)} = A\cdot k\cdot r_1^k + B\cdot k\cdot r_2^k$, respectively. Substituting the particular solution into the recurrence relation (3.12), collecting terms with respect to the two roots and solving for A and B gives $A = (1+\sqrt{3})/3$ and $B = (1-\sqrt{3})/3$. Since $T_L(2,3k) = a^{(h)} + a^{(p)}$, we use the initial conditions $T_L(2,0) = 0$ and $T_L(2,3) = 4$ to obtain $c_1 = 1/(1+\sqrt{3})$ and $c_2 = -1/(1+\sqrt{3})$, which gives Equation (3.7).

To obtain the generating function for the number of straights, we observe that the number of straights in small basic blocks is given by $B_S(2,3) = 2$, $B_S(2,6) = 4$, $B_S(2,9) = 8$, ...and that the number of straights increases by 4 for each increase by 3 in the length of the rectangles being tiled. Therefore, $B_S(2,3k) = (k-1)\cdot 4$ for $k > 1$ and the generating function is given by

(3.13) $$\mathrm{G}_{B_S(2)}(x) = \frac{2x^3(1+x^6)}{(1-x^3)^2}.$$

The generating function for the number of straights in Equation (3.9) now follows from Equation (2.5), together with Equations (3.1) and (3.2). To derive the explicit formula for the number of straights we use Equation (2.3), which relates the total area to the areas covered by Ls and straights, and obtain, after simplification,

(3.14) $$T_S(2,3k) = 2\cdot k\cdot T(2,3k) - T_L(2,3k),$$

from which Equation (3.8) follows. ■

Table 2 gives the first few values for the number of Ls and straights in all tilings of size $2\times n$.

n	3	6	9	12	15	18	21	24
$T_L(2,n)$	4	28	152	744	3436	15284	66224	281424
$T_S(2,n)$	2	16	94	480	2274	10288	193472	815682

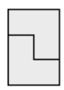
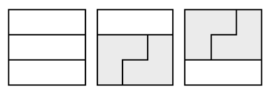
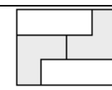
Table 2. The number of Ls and straight in all tromino tilings of $2\times n$ rectangles

Except as mentioned for the sequence in Table 1, none of the sequences in Section 3 are found in Sloane [9].

4. **Tiling** $3\times n$ **Boards**

Counting the number of tilings of the $2\times 3k$ rectangles with trominoes was relatively straightforward, since there are exactly two basic blocks of size $2\times 3k$

for $k > 1$. This resulted in a linear recursion of order two that could be solved explicitly and easily. Unlike the case for the $2 \times n$ rectangles, there is no restriction of the value of n when tiling $3 \times n$ rectangles, since all the areas are automatically divisible by 3. There are basic blocks of all sizes as before, but not a constant number of them. In particular, there is one basic block for $n = 1$, two for $n = 2$, five for $n = 3$; for $n > 3$, there are four basic blocks for $n \equiv 0 \pmod 3$ and two basic blocks for all other values of n. Note that for each basic block of length 4 or more, a block that is three units longer can be constructed by inserting a diagonal of straights through the middle of the given block, as shown in the transition from $n = 4$ to $n = 7$ in Figure 3. The basic blocks of length up to 7 are shown in Figure 3, where the missing blocks are created by reflecting the given ones through the horizontal mid-line.

n	1	2	3	4
				
$B(3,n)$	1	2	5	2

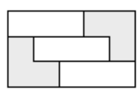
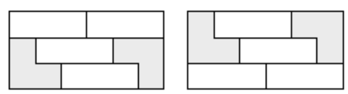
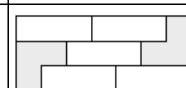
n	5	6	7
			
$B(3,n)$	2	4	2

Figure 3. The basic blocks for tiling $3 \times n$ rectangles with trominoes

Since there is a regular pattern for the number of basic blocks, we are able to obtain a recurrence equation of order six, as well as the generating function for the number of tromino tilings.

Theorem 4.1. *The number of tilings of* $3 \times n$ *rectangles with trominoes is given by the following recursive formula*

(4.1) $$T(3,n) = T(3,n-1) + 2T(3,n-2) + 6T(3,n-3) + T(3,n-4) - T(3,n-6),$$

with $T(3,0) = T(3,1) = 1, T(3,2) = 3$, $T(3,3) = 10$, $T(3,4) = 23$, *and* $T(3,5) = 62$, *and generating function*

(4.2) $$G_{T(3)}(x) = \frac{1}{1 - G_{B(3)}(x)} = \frac{x^3 - 1}{-1 + x + 2x^2 + 6x^3 + x^4 - x^6}.$$

Proof: We use Figure 3 and the discussion preceding it to express $T(3, n)$ recursively using Equation (2.1).

$$T(3,n) = T(3,n-1) + 2 \cdot T(3,n-2) + 5 \cdot T(3,n-3) + 2 \cdot \sum_{i=1}^{\lfloor (n-1)/3 \rfloor} T(3,n-(3i+1)) + 2 \cdot \sum_{i=1}^{\lfloor (n-2)/3 \rfloor} T(3,n-(3i+2)) + 4 \cdot \sum_{i=1}^{\lfloor n/3 \rfloor} T(3,n-3i). \tag{4.3}$$

Since the sums in Equation (4.3) each contain summands referring to tilings whose widths are congruent modulo 3, we can cancel out all but the first terms in each of the sums by computing the difference $T(3,n) - T(3,n-3)$, and then solve for $T(3,n)$. This results in the recurrence of order six given in Equation (4.1). Unlike the case of the $2 \times n$ tromino tilings, this recurrence cannot be factored. The initial conditions for the recurrence follow from Equation (4.3) for $n = 1,\ldots,5$.

The generating function for the basic blocks is derived as follows. All terms show up twice, the multiples of three show up twice more (for a total of 4), and the initial terms for $n = 1$ and $n = 3$ are "manually" adjusted.

$$G_{B(3)}(x) = 2\sum_{i=1}^{\infty} x^i + 2\sum_{i=1}^{\infty} x^{3i} - x + x^3 = \frac{x(1+2x+5x^2+x^3-x^5)}{1-x^3}. \tag{4.4}$$

Then the generating function for the number of tilings follows from Equation (2.2) as before. ■

Table 3 gives the first few values for the number of tilings of $3 \times n$ rectangles with trominoes.

n	0	1	2	3	4	5	6	7	8	9	10
$T(3,n)$	1	1	3	10	23	62	170	441	1173	3127	8266

Table 3. The number of tilings of $3 \times n$ rectangles with trominoes

We next count the number of Ls and straights among all the tilings of $3 \times n$ rectangles with trominoes and obtain the following result.

Theorem 4.2. *The generating functions for the number of Ls and straights in all tilings of* $3 \times n$ *rectangles with trominoes are given by*

(4.5) $$\mathrm{G}_{T_L(3)}(x) = \frac{4x^2(1+x)^2(1-x^3)}{(-1+x+2x^2+6x^3+x^4-x^6)^2}$$

and

(4.6) $$\mathrm{G}_{T_S(3)}(x) = \frac{x+7x^3+2x^4+6x^5+2x^6+3x^7+3x^9}{(-1+x+2x^2+6x^3+x^4-x^6)^2}.$$

Proof: We begin by considering the number of Ls in basic blocks. For $n = 2$ and $n > 3$, the number of Ls is just twice the number of basic blocks. Thus, the generating function for the number of Ls in basic blocks can be obtained from the generating function for basic blocks by making the relevant adjustments for $n = 1$ and $n = 3$, as shown in Equation (4.7).

(4.7) $$\begin{aligned} G_{B_L(3)}(x) &= 2G_{B(3)}(x) - 2x - 2x^3 \\ &= \frac{x(1+2x+5x^2+x^3-x^5)}{1-x^3} - 2x - 2x^3 = \frac{4x^2(1+x)^2}{1-x^3}. \end{aligned}$$

The generating function for the number of Ls in all tilings now follows from Equation (2.7), together with Equations (4.2) and (4.4).

Next we obtain the generating function for the number of straights in basic blocks, which is a little bit more involved. Table 4 shows the number of straights in the basic blocks of size $3 \times n$ for small values of n.

n	1	2	3	4	5	6	7	8	9	10	11
$B_S(3,n)$	1	0	7	4	6	16	10	12	28	16	18

Table 4. The number of straights in tromino tilings of $3 \times n$ rectangles

A regular pattern emerges only for $n \geq 7$, where we find that $B_S(3,n) = B_S(3,n-3) + 3B(3,n-3)$. The total sequence is a combination of three sequences that are all arithmetic. The sequence for multiples of 3 increases by 12 each time, the others increase by 6. Therefore,

(4.8) $$B_s(3,n) = \begin{cases} 16+(k-2)\cdot 12 = (k-1)\cdot 12+4 & \text{if} \quad n = 3k, k \geq 2 \\ 4+(k-1)\cdot 6 = (k-1)\cdot 6+4 & \text{if} \quad n = 3k+1, k \geq 1 \\ 6+(k-1)\cdot 6 = k\cdot 6 & \text{if} \quad n = 3k+2, k \geq 1 \end{cases}$$

with initial conditions as given in Table 4. The corresponding generating function is given by

(4.9)

$$G_{B_S(3)}(x) = 7x^3 + \sum_{k=2}^{\infty}(12(k-1)+4)(x^3)^k + x + \sum_{k=1}^{\infty}(6(k-1)+4)(x^3)^k x + \sum_{k=1}^{\infty} 6k(x^3)^k x^2.$$

Simplifying this expression and applying Equation (2.5) together with Equations (4.2) and (4.4), gives rise to the generating function for the number of straights in all $3 \times n$ tromino tilings. ■

Table 5 gives the number of Ls and straights in all $3 \times n$ tromino tilings for the first few values of n.

n	1	2	3	4	5	6	7	8	9
$T_L(3,n)$	0	4	16	48	172	552	1672	5120	15304
$T_S(3,n)$	1	2	14	44	138	468	1415	12839	37680

Table 5. The number of Ls and straights in all $3 \times n$ tromino tilings

None of the sequences in Section 4 are listed in Sloane [9].

5. Open Questions

The authors are currently exploring questions regarding how many ways one can tile rectangles with tetrominoes. One can also attempt to extend tromino tilings to $4 \times n$ or larger rectangles, although the counting problems become much more complex.

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